Questionable Quadrupling - Solution

We will construct the faces of the initial die iteratively. The key is to note that, if the faces of the die are $f_1 < f_2 < \ldots < f_m$, the minimal quadrupling is $f_1 + f_1 + f_1 + f_1$, and all other possible quadruplings are larger than this. Hence, we can determine f_1, the minimal face on the initial die, by dividing the smallest quadrupling outcome by 4. Of course we can determine f_m in a similar fashion, but it suffices to iterate in only one direction.

Next, we can perform our iteration. Knowing which faces must be on our die, we can compute all possible quadrupling outcomes using only these faces. We then look at differences between the actual list of quadrupling outcomes and our currently computed list. If both are the same, we are clearly done as we have determined a possible list of initial die faces that correspond to the correct quadrupling outcome.

If both are not the same, there are still quadrupling outcomes that are not accounted for with our current die faces, so we must add another face. To compute the value of this face, we can use a similar idea to how we determined f_1. Say that f_i is the smallest face, not yet discovered. Certainly all of the quadrupling outcomes that are unaccounted for, must have at least one of the four dice rolls be one of the undiscovered faces (as we account for all other quadrupling outcomes already). The smallest possible unaccounted quadrupling must thus be $f_i + f_1 + f_1 + f_1$, as there are no smaller undiscovered faces than f_i, and f_1 is the smallest face on the entire initial die. Thus, as we already know f_1 from the first step, we can compute f_i by subtracting $3f_1$ from the smallest unaccounted quadrupling outcome. Iteratively we will now eventually discover all of the faces. As f_i was the smallest unknown die face, we in fact discover the faces in increasing order, which was also the order in which they should be output. Hence, we do not need to sort the output!

A final note should be made on the computation of the possible quadrupling outcomes with the currently known initial die faces. The easiest approach would be to simply iterate over all possible tuples of four rolls and recompute the entire list. As the variables are quite small, this should be fast enough. However, we can do slightly better by updating the list of quadrupling outcomes. After adding f_i, we know that all the new quadrupling outcomes that are possible, must have at least one f_i. Hence, we only need to iterate over all possible tuples over 3 rolls, for the other three rolls. We should keep in mind that the f_i could be any of the four rolls, and thus need to multiply by a binomial coefficient. See the solution code for specifics on this implementation.